

# Superfield Lagrangian Quantization with Extended BRST Symmetry

P.M. LAVROV\*, P.YU. MOSHIN

*Tomsk State Pedagogical University, Tomsk 634041, Russia*

We consider possible superfield representations of extended BRST symmetry for general gauge theories within the principle of gauge-fixing based on a generating equation for the gauge functional. We examine admissible superfield choices for an extended antibracket and delta-operator with given algebraic properties and show that only one of these choices is compatible with the requirement of extended BRST symmetry realized in terms of supertranslations along Grassmann coordinates. We demonstrate that this realization leads to the gauge-independence of the  $S$ -matrix.

## 1. Introduction

In the past few years two tendencies have made themselves manifest in the development of covariant quantization methods based on (extended) BRST symmetry [1, 2]: one is introduction of gauge with the help of equations imposed on gauge-fixing functionals [3, 4, 5]; the other is formulation of superfield quantization rules [6, 7, 8, 9]. Recently both these concepts have been combined within the modified superfield formalism [10], which generalizes the previous studies [3, 6] dealing with different modifications of the BV quantization [11] for general gauge theories in the framework of the standard BRST symmetry [1].

In [3], it was shown that the BV approach can be formulated in such a way that the entire gauge-fixing part of the quantum action is subject to a special generating equation analogous to the usual generating equation that determines the quantum action. It was demonstrated that this concept of gauge-fixing guarantees the independence of the vacuum functional under infinitesimal gauge variations, which, by virtue of the equivalence theorem [12], implies the gauge-independence of the  $S$ -matrix.

In [6], a superfield form of the BV quantization rules was proposed. The variables of the BV approach were combined into superfields  $\Phi^A(\theta)$  and superantifields  $\Phi_A^*(\theta)$  with opposite Grassmann parities, defined in a superspace with one anticommuting coordinate  $\theta$ . A superfield representation of the usual antibracket  $( , )$  and the operator  $\Delta$  was found. The vacuum functional is given as a functional integral over  $\Phi^A(\theta)$  and  $\Phi_A^*(\theta)$ . The components of superfields and superantifields are interpreted as the usual field-antifield variables  $\phi^A$ ,  $\phi_A^*$  as well as the Langrange multipliers  $\lambda^A$  and the sources  $J_A$  to the fields. The fact that the components of the supervariables play different roles in the BV approach leads to a difference in the treatment of  $\Phi^A(\theta)$  and  $\Phi_A^*(\theta)$ . Thus, to obtain the correspondence with the BV vacuum functional one has to impose the constraint  $J_A = 0$  which is introduced in a manifestly superfield form by adding an integration density depending on  $\Phi_A^*(\theta)$ . In the space of the supervariables, the transformations of BRST symmetry are realized in terms of variations induced by supertranslations along

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\*E-mail: lavrov@tspu.edu.ru

the Grassmann coordinate. The generators of supertranslations naturally possess the properties of nilpotency and anticommutativity with each other as well as with the delta-operator, and their properties of differentiating the antibracket are identical with the well-known property of  $\Delta$ . The superfield BRST transformations were shown to encode the gauge-independence of the vacuum functional and, consequently, that of the corresponding  $S$ -matrix.

Finally, in [10] the superfield formalism [6] was modified in such a way that the gauge-fixing quantum action  $X$  in the vacuum functional was required to satisfy a special generating equation,

$$\frac{1}{2}(X, X) - UX = i\hbar\Delta X, \quad (1.1)$$

expressed in terms of the generator  $U$  of supertranslations in the space of superfields  $\Phi^A(\theta)$ , and formally analogous to the equation [6] for the quantum action  $W$

$$\frac{1}{2}(W, W) + VW = i\hbar\Delta W, \quad (1.2)$$

containing the generator  $V$  of supertranslations in the space of superantifields  $\Phi_A^*(\theta)$ . The above generating equations (1.1) and (1.2) make it possible to encode the gauge-independence of the  $S$ -matrix in terms of the transformations of BRST symmetry, realized as a combination of supertranslations along the Grassmann coordinate and anticanonical transformations generated by the antibracket,

$$\begin{aligned} \delta\Phi^A(\theta) &= \mu U\Phi^A(\theta) + (\Phi^A(\theta), X - W)\mu, \\ \delta\Phi_A^*(\theta) &= \mu V\Phi_A^*(\theta) + (\Phi_A^*(\theta), X - W)\mu, \end{aligned} \quad (1.3)$$

where  $\mu$  is a constant anticommuting parameter. The modified superfield formalism [10] is equivalent to the gauge-fixing procedure [3] and contains the original superfield approach [6] as a special case of solutions to the generating equation for the gauge functional  $X$ .

Note that there exists an alternative approach [7] to superfield quantization of gauge theories, following from a superfield Hamiltonian formalism [7] for constrained dynamical systems in the framework of BRST symmetry.

Consider the most important differences between the methods [7] and [6, 10]. First, the set of supervariables required for the construction of the vacuum functional in the covariant formulation [7] is composed by field-antifield pairs<sup>1</sup>  $\Phi^A(\theta)$ ,  $\Phi_A^*(\theta)$  with the same Grassmann parity. Second, the method [7] is formally a *direct* extension of the BV approach, with the field-antifield variables replaced by their superfield counterparts. As a consequence, at the component level, the corresponding antibracket and delta-operator [7] are extensions of the original BV objects, constructed with the help of the superpartners  $\lambda^A$ ,  $J_A$  of the usual field-antifield variables  $\phi^A$ ,  $\phi_A^*$ . As another consequence, the vacuum functional is given by a functional integral over the superfields  $\Phi^A(\theta)$  with the variables  $\Phi_A^*(\theta)$  placed on a specified gauge hypersurface.

Although essentially different at the superfield level, the proposals [6, 7, 10] are equivalent to each other in the sense that they parameterize the BV vacuum functional after imposing appropriate restrictions on the solutions of the generating equations.

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<sup>1</sup>We assume the separation of variables in terms of Darboux coordinates. For illustrative purposes, we have replaced the original notations [7] by those of [6], also in terms of the usual field-antifield variables  $\phi^A$ ,  $\phi_A^*$  and their superpartners  $\lambda^A$ ,  $J_A$ , despite these latter do not admit of the interpretation [6] from the viewpoint of the BV method. Note that the variables  $\Phi_A^*(\theta)$  considered in [7] are *formally* related to those introduced in [6] by transposition of the components  $\phi_A^*$  and  $J_A$ .

As in the case of the standard BRST symmetry realized in the reviewed papers [3, 6], there have been studies [4, 5, 8] devoted to the implementation of these concepts of gauge-fixing [3] and superfield quantization [6] on the basis of extended BRST symmetry [2].

In the framework of the triplectic [4] and modified triplectic [5] methods, the idea of imposing a generating equation on the gauge-fixing part of the quantum action was incorporated into the  $Sp(2)$  covariant formulation [13] of extended BRST symmetry for general gauge theories. One of the ingredients of the triplectic (also called “completely anticanonical”) approach is to consider part of the antifields and Lagrange multipliers of the  $Sp(2)$  covariant formalism [13] as anticanonically conjugated variables, which involves the corresponding redefinition of the original [13] extended antibracket and delta-operator, retaining, however, their essential algebraic properties.

In [8], a superfield form of the  $Sp(2)$  covariant quantization rules was proposed along the lines of the papers [6]. The variables of the  $Sp(2)$  covariant approach were combined into a set of superfields  $\Phi^A(\theta)$  and supersources  $\bar{\Phi}_A(\theta)$ , with the same Grassmann parity, defined in a superspace with two anticommuting coordinates  $\theta^a$ . The components of the superfields and supersources are interpreted as the fields  $\phi^A$ , the antifields  $\phi_{Aa}^*$ ,  $\bar{\phi}_A$ , the Lagrange multipliers  $\pi^{Aa}$ ,  $\lambda^A$ , and the sources  $J_A$ . The transformations of extended BRST symmetry are realized in terms of supertranslations in superspace with the corresponding generators  $U^a$  (in the space of superfields) and  $V^a$  (in the space of supersources) possessing the properties of generalized nilpotency and anticommutativity. The algebraic properties of the operators  $U^a$  and  $V^a$  with respect to the extended antibracket  $( , )^a$  and the operator  $\Delta^a$  naturally generalize the corresponding properties of the operators  $U$  and  $V$  with respect to the objects  $( , )$  and  $\Delta$  of the superfield quantization rules [6, 10].

As compared to the superfield procedures [6, 10], where the component representation of the antibracket and delta-operator is identical with the corresponding objects of the BV formalism, the superfield procedure [8] applies an extended antibracket  $( , )^a$  and operator  $\Delta^a$  whose component representation turns out to coincide with the objects of the completely anticanonical approach [4, 5], rather than with their original counterparts of the  $Sp(2)$  covariant scheme [13].

Note that within the superfield methods [6, 10], based on the standard BRST symmetry, where the superspace contains a single anticommuting coordinate, there exists only one possibility of constructing the antibracket  $( , )$  and the operator  $\Delta$  with the algebraic properties [6, 10], under such natural requirements as a specific Grassmann parity and locality in  $\theta$ . In this sense, there is a unique realization of the superfield rules [6, 10], given the set of supervariables, the general form of the vacuum functional and the generating equations.

In the case of a superfield formalism based on the extended BRST symmetry, the situation appears to be more complicated, because a superspace with two Grassmann coordinates admits different possibilities of constructing objects with the properties [13] of the extended antibracket  $( , )^a$  and the operator  $\Delta^a$ . Thus, in the recent paper [9], a superfield form of the  $osp(1, 2)$  covariant quantization rules [14] was proposed, where another realization of superfield objects with the given properties was found. Moreover, the component form of the these objects is identical with the original extended antibracket  $( , )^a$  and the operator  $\Delta^a$ , as defined in [13].

In this paper we investigate possible superfield representations of extended BRST symmetry for general gauge theories within the principle of gauge-fixing based on a generating equation for the gauge functional.

To this end, we generalize the superfield  $Sp(2)$  covariant scheme [8] along the lines of the modified superfield approach [10], taking into account the arbitrariness in a specific choice of its basic objects. Thus, we postulate generating equations for the superfiled quantum action  $S(\Phi, \bar{\Phi})$  and the gauge-fixing functional  $X(\Phi, \bar{\Phi})$  in a form analogous to eqs. (1.1) and (1.2), i.e. expressed in terms of an extended antibracket  $( , )^a$  and operators

$\Delta^a$ ,  $U^a$ ,  $V^a$ . While the manifest form of the operators  $U^a$  and  $V^a$  is determined by their interpretation as generators of supertranslations, there is still an ambiguity in the choice of the extended antibracket  $( , )^a$  and the operator  $\Delta^a$  with the given algebraic properties [8].

In this connection, we consider possible representations of these objects, and observe that they are reduced to two admissible choices, whose component form in one case coincides with the extended antibracket and delta-operator of the original  $Sp(2)$  covariant scheme [13], and in the other case, with their counterparts of the triplectic approach [4, 5]. Defining the vacuum functional as a straightforward, in the sense of [10], extension of the vacuum functional [8], we demonstrate that only one of the two choices for the objects  $( , )^a$  and  $\Delta^a$ , namely, the one applied by the completely anticanonical procedure [4, 5], is compatible with the requirement of extended BRST symmetry considered as a generalization of eqs. (1.3) in the form of variations of superfields  $\Phi^A(\theta)$  and supersources  $\bar{\Phi}_A(\theta)$  induced by supertranslations combined with anticanonical transformations generated by the extended antibracket  $( , )^a$ . Furthermore, we show that the postulated form of extended BRST symmetry encodes the gauge-independence of the  $S$ -matrix.

We demonstrate that, on the one hand, the resulting quantization rules provide a superfield form of the modified triplectic approach [5], and, on the other hand, they contain the superfield  $Sp(2)$  covariant scheme [8] as a particular case of solutions for the gauge-fixing functional  $X$ .

The paper is organized as follows. In Section 2 we introduce the basic objects  $U^a$ ,  $V^a$ ,  $\Delta^a$ ,  $( , )^a$  with the algebraic properties [8] and give their manifest superfield representation. In Section 3 we extend the quantization rules [8] along the lines of [10] and determine the choice of  $\Delta^a$  and  $( , )^a$  compatible with the given form of extended BRST symmetry. In Section 4 we demonstrate the gauge-independence of the  $S$ -matrix in the proposed superfield formalism. In Section 5 we discuss the connection of our formalism with the methods [5, 8]. In Appendix we analyze the properties of the operators  $U^a$ ,  $V^a$  and  $\Delta^a$  from the viewpoint of Lie superalgebras.

We use the condensed notations [15] and the conventions adopted in [8].

## 2. Main Definitions

Consider a superspace  $(x^\mu, \theta^a)$ , where  $x^\mu$  are space-time coordinates, and  $\theta^a$  is an  $Sp(2)$  doublet of anticommuting coordinates. Note that any function  $f(\theta)$  has a component representation,

$$f(\theta) = f_0 + \theta^a f_a + \theta^2 f_3, \quad \theta^2 \equiv \frac{1}{2} \theta_a \theta^a,$$

and an integral representation,

$$f(\theta) = \int d^2 \theta' \delta(\theta' - \theta) f(\theta'), \quad \delta(\theta' - \theta) = (\theta' - \theta)^2,$$

where raising and lowering the  $Sp(2)$  indices is performed by the rule  $\theta^a = \varepsilon^{ab} \theta_b$ ,  $\theta_a = \varepsilon_{ab} \theta^b$ , with  $\varepsilon^{ab}$  being a constant antisymmetric tensor,  $\varepsilon^{12} = 1$ , and integration over  $\theta^a$  is given by

$$\int d^2 \theta = 0, \quad \int d^2 \theta \theta^a = 0, \quad \int d^2 \theta \theta^a \theta^b = \varepsilon^{ab}.$$

In particular, for any function  $f(\theta)$  we have

$$\int d^2 \theta \frac{\partial f(\theta)}{\partial \theta^a} = 0,$$

which implies the property of integration by parts

$$\int d^2\theta \frac{\partial f(\theta)}{\partial \theta^a} g(\theta) = - \int d^2\theta (-1)^{\varepsilon(f)} f(\theta) \frac{\partial g(\theta)}{\partial \theta^a},$$

where derivatives with respect to  $\theta^a$  are taken from the left.

According to [8], we now introduce a set of superfields  $\Phi^A(\theta)$ ,  $\varepsilon(\Phi^A) = \varepsilon_A$ , with the boundary condition

$$\Phi^A(\theta)|_{\theta=0} = \phi^A$$

and a set of supersources  $\bar{\Phi}_A(\theta)$  of the same Grassmann parity,  $\varepsilon(\bar{\Phi}_A) = \varepsilon_A$ .

Denote by  $U^a$  and  $V^a$  doublets of Fermionic operators [8] generating transformations of superfields and supersources induced by supertranslations  $\theta^a \rightarrow \theta^a + \mu^a$  along the Grassmann coordinates,

$$\begin{aligned}\delta\Phi^A(\theta) &= \mu_a \frac{\partial\Phi^A(\theta)}{\partial\theta_a} = \mu_a U^a \Phi^A(\theta), \\ \delta\bar{\Phi}_A(\theta) &= \mu_a \frac{\partial\bar{\Phi}_A(\theta)}{\partial\theta_a} = \mu_a V^a \bar{\Phi}_A(\theta).\end{aligned}$$

The generators  $U^a$  and  $V^a$  can be represented as first-order differential operators, having the form of  $\theta$ -local functionals,

$$\begin{aligned}U^a &= \int d^2\theta \frac{\partial\Phi^A(\theta)}{\partial\theta_a} \frac{\delta_l}{\delta\Phi^A(\theta)}, \\ V^a &= \int d^2\theta \frac{\partial\bar{\Phi}_A(\theta)}{\partial\theta_a} \frac{\delta}{\delta\bar{\Phi}_A(\theta)},\end{aligned}\tag{2.1}$$

where

$$\begin{aligned}\frac{\delta_l\Phi^A(\theta)}{\delta\Phi^B(\theta')} &= \delta(\theta' - \theta)\delta_B^A = \frac{\delta\Phi^A(\theta)}{\delta\Phi^B(\theta')}, \\ \frac{\delta\bar{\Phi}_A(\theta)}{\delta\bar{\Phi}_B(\theta')} &= \delta(\theta' - \theta)\delta_A^B.\end{aligned}$$

From eqs. (2.1) follow the algebraic properties

$$U^{\{a}U^{b\}} = 0, \quad V^{\{a}V^{b\}} = 0, \quad V^aU^b + U^bV^a = 0.\tag{2.2}$$

Let us introduce a doublet of Fermionic second-order differential operators  $\Delta^a$  with the properties of generalized nilpotency and anticommutativity [8]

$$\begin{aligned}\Delta^{\{a}\Delta^{b\}} &= 0, \\ \Delta^{\{a}V^{b\}} + V^{\{a}\Delta^{b\}} &= 0, \\ \Delta^{\{a}U^{b\}} + U^{\{a}\Delta^{b\}} &= 0,\end{aligned}\tag{2.3}$$

where the curly brackets denote symmetrization over  $Sp(2)$  indices,  $A^{\{a}B^{b\}} = A^aB^b + A^bB^a$ .

Note that the relations (2.2), (2.3) imposed on arbitrary linearly independent Fermionic doublets  $U^a$ ,  $V^a$ ,  $\Delta^a$  define a set of nilpotent Lie superalgebras  $\mathcal{G}$ , with  $6 \leq \dim \mathcal{G} \leq 8$  (see Appendix).

The action of  $\Delta^a$  on the product of any two functionals  $F, G$  defines an antibracket operation  $(\ , \ )^a$

$$\Delta^a(F \cdot G) = (\Delta^a F) \cdot G + F \cdot (\Delta^a G)(-1)^{\varepsilon(F)} + (F, G)^a(-1)^{\varepsilon(F)} \quad (2.4)$$

with the properties

$$\begin{aligned} \varepsilon((F, G)^a) &= \varepsilon(F) + \varepsilon(G) + 1, \\ (F, G)^a &= -(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}(G, F)^a, \\ D^{\{a}(F, G)^b\}} &= (D^{\{a}F, G)^b\}} - (F, D^{\{a}G)^b\}}(-1)^{\varepsilon(F)}, \end{aligned} \quad (2.5)$$

$$(F, GH)^a = (F, G)^a H + (F, H)^a G(-1)^{\varepsilon(G)\varepsilon(H)}, \quad (2.6)$$

$$((F, G)^{\{a}, H)^b\})(-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \text{cycle } (F, G, H) \equiv 0, \quad (2.7)$$

where  $D^a = (\Delta^a, U^a, V^a)$ . Note that eqs. (2.5) follow immediately from the definition (2.4) and the relations (2.3). Eq. (2.6) is the consequence of the fact that  $\Delta^a$  in eq. (2.4) is assumed to be a second-order differential operator, while the generalized Jacobi identity (2.7) follows from eqs. (2.3)–(2.6).

Finally, in terms of  $U^a, V^a$  and  $\Delta^a$  we define the operators

$$\bar{\Delta}^a = \Delta^a + \frac{i}{\hbar}V^a, \quad \tilde{\Delta}^a = \Delta^a - \frac{i}{\hbar}U^a \quad (2.8)$$

with the properties

$$\bar{\Delta}^{\{a}\bar{\Delta}^{b\}} = 0, \quad \tilde{\Delta}^{\{a}\tilde{\Delta}^{b\}} = 0, \quad \bar{\Delta}^{\{a}\tilde{\Delta}^{b\}} + \tilde{\Delta}^{\{a}\bar{\Delta}^{b\}} = 0$$

and

$$\begin{aligned} \bar{\Delta}^{\{a}(F, G)^b\}} &= (\bar{\Delta}^{\{a}F, G)^b\}) - (F, \bar{\Delta}^{\{a}G)^b\})(-1)^{\varepsilon(F)}, \\ \tilde{\Delta}^{\{a}(F, G)^b\}} &= (\tilde{\Delta}^{\{a}F, G)^b\}) - (F, \tilde{\Delta}^{\{a}G)^b\})(-1)^{\varepsilon(F)}, \end{aligned}$$

following from eqs. (2.2), (2.3) and (2.5).

An explicit form of the extended operator  $\Delta^a$  and the corresponding extended antibracket  $(\ , \ )^a$  with the given properties is not unique.

Consider the class of Fermionic second-order differential operators (in the space of superfields and supersources) such that the dependence on the components of the  $\Phi^A(\theta)$  and  $\bar{\Phi}_A(\theta)$  enters only through the derivatives

$$\frac{\delta}{\delta\Phi^A(\theta)}, \quad \frac{\delta}{\delta\bar{\Phi}_A(\theta)}.$$

In the specified class there exist only two linearly independent  $Sp(2)$  doublets having the form of  $\theta$ -local functionals and possessing the algebraic properties (2.3) of the extended delta-operator.<sup>2</sup> Due to an additional property of anticommutativity with each other,

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<sup>2</sup>In all, the specified class contains four linearly independent  $Sp(2)$  doublets having the form of  $\theta$ -local operator functionals. The possibilities are limited to integrands constructed from the functional derivatives  $\frac{\delta}{\delta\Phi^A(\theta)}$ ,  $\frac{\delta}{\delta\bar{\Phi}_A(\theta)}$  and different combinations of  $\theta^a$ ,  $\frac{\partial}{\partial\theta^a}$ . Note that the analysis of such combinations is simplified due to integration by parts and the use of the anticommutator  $\{\theta^a, \frac{\partial}{\partial\theta^b}\} = \delta_b^a$ .

these operators span a two-dimensional linear space of operators with the properties of  $\Delta^a$ . The basis elements  $\Delta_1^a$ ,  $\Delta_2^a$  of this space can be chosen in the form

$$\Delta_1^a = - \int d^2\theta \frac{\delta_l}{\delta\Phi^A(\theta)} \frac{\partial}{\partial\theta_a} \frac{\delta}{\delta\bar{\Phi}_A(\theta)}, \quad (2.9)$$

$$\Delta_2^a = \int d^2\theta \frac{\delta_l}{\delta\Phi^A(\theta)} \frac{\partial^2}{\partial\theta^2} \left( \theta^a \frac{\delta}{\delta\bar{\Phi}_A(\theta)} \right), \quad (2.10)$$

where

$$\frac{\partial^2}{\partial\theta^2} \equiv \frac{1}{2} \varepsilon^{ab} \frac{\partial}{\partial\theta^b} \frac{\partial}{\partial\theta^a}.$$

The extended delta-operators (2.9) and (2.10) generate the corresponding extended antibrackets

$$(F, G)_1^a = \int d^2\theta \left\{ \frac{\delta F}{\delta\Phi^A(\theta)} \frac{\partial}{\partial\theta_a} \frac{\delta G}{\delta\bar{\Phi}_A(\theta)} (-1)^{\varepsilon_A+1} - (F \leftrightarrow G) (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} \right\} \quad (2.11)$$

and

$$(F, G)_2^a = \int d^2\theta \left\{ \left( \frac{\partial^2}{\partial\theta^2} \frac{\delta F}{\delta\Phi^A(\theta)} \right) \theta^a \frac{\delta G}{\delta\bar{\Phi}_A(\theta)} (-1)^{\varepsilon_A} - (F \leftrightarrow G) (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} \right\}. \quad (2.12)$$

The choice of the operator  $\Delta^a$  and the antibracket  $( , )^a$  in the form (2.9), (2.11) is identical with the one used in [8], whereas the other choice, in the form (2.10), (2.12), was made in [9].

Despite the fact that the specific representations (2.9), (2.10) of the operator  $\Delta^a$  both satisfy eqs. (2.3), they are nevertheless not equivalent at the algebraic level. Namely, it can be shown that these choices select two different Lie superalgebras associated with the whole set of relations (2.2), (2.3) for the operators  $U^a$ ,  $V^a$ ,  $\Delta^a$  (see Appendix).

### 3. Quantization Rules. Extended BRST Symmetry

Consider a generalization of the superfield  $Sp(2)$  covariant quantization rules [8] modified along the lines of [10]. Define the vacuum functional  $Z$  as the following path integral:

$$Z = \int d\Phi d\bar{\Phi} \rho(\bar{\Phi}) \exp \left\{ \frac{i}{\hbar} \left[ W(\Phi, \bar{\Phi}) + X(\Phi, \bar{\Phi}) + \bar{\Phi}\Phi \right] \right\}, \quad (3.1)$$

where  $W = W(\Phi, \bar{\Phi})$  is a quantum action that satisfies the generating equation

$$\bar{\Delta}^a \exp \left\{ \frac{i}{\hbar} W \right\} = 0, \quad (3.2)$$

and  $X = X(\Phi, \bar{\Phi})$  is a Bosonic gauge-fixing functional subject to the equation

$$\tilde{\Delta}^a \exp \left\{ \frac{i}{\hbar} X \right\} = 0, \quad (3.3)$$

with  $\bar{\Delta}^a$  and  $\tilde{\Delta}^a$  defined by eqs. (2.8). Eqs. (3.2) and (3.3) are equivalent to

$$\frac{1}{2}(W, W)^a + V^a W = i\hbar\Delta^a W, \quad (3.4)$$

$$\frac{1}{2}(X, X)^a - U^a X = i\hbar\Delta^a X. \quad (3.5)$$

In eq. (3.1), we have used the notation  $\rho(\bar{\Phi})$  for a functional which defines the weight of integration over the supersources  $\bar{\Phi}_A(\theta)$  and has the form of a functional  $\delta$  function,

$$\rho(\bar{\Phi}) = \delta \left( \int d^2\theta \bar{\Phi}(\theta) \right). \quad (3.6)$$

We have also introduced the functional

$$\bar{\Phi}\Phi \equiv \int d^2\theta \bar{\Phi}_A(\theta)\Phi^A(\theta). \quad (3.7)$$

Define the transformations of extended BRST symmetry as the following transformations of global supersymmetry:

$$\begin{aligned} \delta\Phi^A(\theta) &= \mu_a U^a \Phi^A(\theta) + (\Phi^A(\theta), X - W)^a \mu_a, \\ \delta\bar{\Phi}_A(\theta) &= \mu_a V^a \bar{\Phi}_A(\theta) + (\bar{\Phi}_A(\theta), X - W)^a \mu_a. \end{aligned} \quad (3.8)$$

On the one hand, eqs. (3.8) provide a straightforward generalization of eqs. (1.3) introduced in [10]. One the other hand, the choice of extended BRST symmetry in this particular form is motivated by the fact that in case the transformations (3.8) do realize the invariance of the integrand (3.1), then in combination with the generating equations (3.2) and (3.3) they also guarantee the gauge-independence of the vacuum functional.

Note that prior to introducing the symmetry transformations (3.8) the explicit choice of the extended antibracket  $( , )^a$  in the form (2.11) has no clear advantage over the other choice (2.12). Let us show, however, that eq. (2.11) does meet the above requirement of extended BRST symmetry, while eq. (2.12) does not.

Consider the change of the integrand in eq. (3.1) under the transformations (3.8). To examine the change of the exponential in eq. (3.1), note that the variation of an arbitrary functional  $F$  under the transformations

$$\delta_{(1)}\Phi^A(\theta) = \mu_a U^a \Phi^A(\theta), \quad \delta_{(1)}\bar{\Phi}_A(\theta) = \mu_a V^a \bar{\Phi}_A(\theta),$$

induced by supertranslations, has the form

$$\delta_{(1)}F = \mu_a (U^a + V^a)F.$$

In particular, for the functional  $\bar{\Phi}\Phi$  (3.7) we have

$$\delta_{(1)}(\bar{\Phi}\Phi) = 0.$$

At the same time, in both cases (2.11) and (2.12) of explicit representation of the extended antibracket the anticanonical transformations

$$\delta_{(2)}\Phi^A(\theta) = (\Phi^A(\theta), Y)^a \mu_a, \quad \delta_{(2)}\bar{\Phi}_A(\theta) = (\bar{\Phi}_A(\theta), Y)^a \mu_a, \quad \varepsilon(Y) = 0$$

generate the corresponding transformations of an arbitrary functional  $F$

$$\delta_{(2)}F = (F, Y)^a \mu_a.$$

As a result, eqs. (3.8) lead to the variation  $\delta = \delta_{(1)} + \delta_{(2)}$

$$\begin{aligned} \delta(W + X + \bar{\Phi}\Phi) &= \mu_a \left( (W, W)^a - (X, X)^a + (U^a + V^a)(W + X) \right) + \\ &+ \mu_a (\bar{\Phi}\Phi, W - X)^a. \end{aligned} \quad (3.9)$$

To examine the change of the integration measure in eq. (3.1), note that in both cases (2.11) and (2.12) the weight functional  $\rho(\bar{\Phi})$  is invariant under the transformations (3.8),  $\delta\rho(\bar{\Phi}) = 0$ , while the corresponding Jacobian  $J$  has the form

$$J = \exp(2\mu_a \Delta^a W - 2\mu_a \Delta^a X). \quad (3.10)$$

Denote by  $I$  the integrand in eq. (3.1). Then, by virtue of eqs. (3.4), (3.5), (3.9) and (3.10), its variation  $\delta I$  under the transformations (3.8) is given by

$$\delta I = i\hbar^{-1} \mu_a I \left( (U^a - V^a)(W - X) + (\bar{\Phi}\Phi, W - X)^a \right), \quad (3.11)$$

and hence the condition of invariance of the integrand takes the form

$$(U^a - V^a)(W - X) + (\bar{\Phi}\Phi, W - X)^a = 0. \quad (3.12)$$

The fulfillment of eq. (3.12) obviously depends on a specific choice of the extended antibracket. Thus in the case of the antibracket (2.11) the above condition is satisfied due to the identity

$$(\bar{\Phi}\Phi, F)^a = (V^a - U^a)F,$$

which, according to eq. (3.11), implies the invariance of the integrand in eq. (3.1) under the transformations (3.8).

On the other hand, in the case of the antibracket (2.12) we have

$$(\bar{\Phi}\Phi, F)^a \not\equiv (V^a - U^a)F,$$

which means that eq. (3.12) does not hold identically, and therefore the integrand is not invariant under the transformations (3.8) without additional restrictions on the functionals  $W$  and  $X$ .

## 4. Gauge Independence

Let us study the dependence of the vacuum functional  $Z$  (3.1) on the choice of gauge, using the explicit form of the extended antibracket (2.11) and the corresponding extended delta-operator (2.9), which provide the invariance of the integrand (3.1) under the transformations (3.8).

Note, first of all, that any admissible variation  $\delta X$  of the gauge functional  $X$  must satisfy the equation

$$(X, \delta X)^a - U^a \delta X = i\hbar \Delta^a \delta X,$$

which can be represented in the form

$$\hat{Q}^a(X) \delta X = 0. \quad (4.1)$$

In eq. (4.1), we have introduced an operator  $\hat{Q}^a(X)$  possessing the property of generalized nilpotency,

$$\hat{Q}^a(X) = \hat{\mathcal{B}}^a(X) - i\hbar \tilde{\Delta}^a, \quad \hat{Q}^{\{a}(X) \hat{Q}^{b\}}(X) = 0, \quad (4.2)$$

where  $\hat{\mathcal{B}}^a(X)$  stands for an operator acting by the rule

$$(X, F)^a \equiv \hat{\mathcal{B}}^a(X)F$$

and possessing the property

$$\hat{\mathcal{B}}^{\{a}(X)\hat{\mathcal{B}}^{b\}}(X) = \hat{\mathcal{B}}^{\{a}\left(\frac{1}{2}(X, X)^{b\}}\right).$$

By virtue of the operator  $\hat{Q}^a(X)$  in eqs. (4.2), any functional

$$\delta X = \frac{1}{2}\varepsilon_{ab}\hat{Q}^a(X)\hat{Q}^b(X)\delta F, \quad (4.3)$$

parameterized by an arbitrary Boson  $\delta F$ , satisfies eq. (4.1). Furthermore, by analogy with the theorems proved in [13], it can be established that any solution of the equations (4.1), vanishing when all the variables entering the functional  $\delta X$  are equal to zero, has the form (4.3) with a certain Bosonic functional  $\delta F$ .

Denote by  $Z_X \equiv Z$  the value of the vacuum functional (3.1) corresponding to the choice of gauge condition in the form of the functional  $X$ . In the vacuum functional  $Z_{X+\delta X}$  we first make the change of variables (3.8) with  $\mu_a = \mu_a(\Phi, \bar{\Phi})$ , and then the additional change of variables

$$\delta\Phi^A = (\Phi^A, \delta Y_a)^a, \quad \delta\bar{\Phi}_A = (\bar{\Phi}_A, \delta Y_a)^a, \quad \varepsilon(\delta Y_a) = 1$$

with  $\delta Y_a = -i\hbar\mu_a(\Phi, \bar{\Phi})$ . We get

$$Z_{X+\delta X} = \int d\Phi d\bar{\Phi} \rho(\bar{\Phi}) \exp \left\{ \frac{i}{\hbar} \left( W + X + \delta X + \delta X_1 + \bar{\Phi}\Phi \right) \right\},$$

where

$$\delta X_1 = 2 \left( (X, \delta Y_a)^a - U^a \delta Y_a - i\hbar\Delta^a \delta Y_a \right) = 2\hat{Q}^a(X)\delta Y_a.$$

Having eq. (4.3) in mind, choose the functional  $\delta Y_a$  in the form

$$\delta Y_a = -\frac{1}{4}\varepsilon_{ab}\hat{Q}^b(X)\delta F.$$

Then we find that  $\delta X + \delta X_1 = 0$  and conclude that the relation  $Z_{X+\delta X} = Z_X$  holds true. This implies that the symmetry transformations (3.8) do encode the gauge-independence of the  $S$ -matrix within the proposed superfield formalism, and therefore they play the role of the transformations of extended BRST symmetry.

## 5. Discussion

In this paper we have extended the  $Sp(2)$  covariant superfield approach [8] to general gauge theories on the basis of fixing the gauge in terms of a special generating equation (3.3) imposed on the gauge functional. We have observed that the possibilities of explicit representation of the formalism in terms of the operator  $\Delta^a$  and the extended antibracket  $(\ , \ )^a$  with the given algebraic properties [8] are reduced to two different choices [8, 9]. We have shown that only one of the two possibilities, in fact [8], is compatible with the requirement of extended BRST symmetry (3.8) realized in terms of supertranslations along the Grassmann coordinates of the superspace. We have demonstrated that this form of symmetry transformations ensures the gauge-independence of the  $S$ -matrix.

On the one hand, the quantization rules based on the manifest form of the operator  $\Delta^a$  and the extended antibracket [8] actually contain the superfield  $Sp(2)$  covariant scheme [8] as a particular case of gauge-fixing. Indeed, any functional

$$X(\Phi) = \frac{1}{2}\varepsilon_{ab}U^aU^bF(\Phi),$$

parameterized by an arbitrary Boson,  $F = F(\Phi)$ , is a solution of the generating equation (3.5) and represents the exact form of the gauge functional used in [8].

On the other hand, the proposed method can be considered as a superfield form of the modified triplectic approach suggested in [5]. Indeed, consider the component representation of superfields  $\Phi^A(\theta)$  and supersources  $\bar{\Phi}_A(\theta)$

$$\begin{aligned}\Phi^A(\theta) &= \phi^A + \pi^{Aa}\theta_a + \frac{1}{2}\lambda^A\theta_a\theta^a, \\ \bar{\Phi}_A(\theta) &= \bar{\phi}_A - \theta^a\phi_{Aa}^* - \frac{1}{2}\theta_a\theta^a J_A.\end{aligned}$$

The set of variables  $(\phi^A, \pi^{Aa}, \lambda^A, \phi_{Aa}^*, \bar{\phi}_A, J_A)$  is identical with the sets of variables applied by the  $Sp(2)$  covariant [13], triplectic [4] and modified triplectic [5] quantization schemes.

Denote  $F(\Phi, \bar{\Phi}) \equiv \tilde{F}(\phi, \pi, \lambda, \bar{\phi}, \phi^*, J)$ . Then the component representation of the extended antibracket (2.11)

$$(F, G)^a = \frac{\delta \tilde{F}}{\delta \phi^A} \frac{\delta \tilde{G}}{\delta \phi_{Aa}^*} + \varepsilon^{ab} \frac{\delta \tilde{F}}{\delta \pi^{Ab}} \frac{\delta \tilde{G}}{\delta \bar{\phi}_A} - (\tilde{F} \leftrightarrow \tilde{G}) (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}$$

and the operator  $\Delta^a$  (2.9)

$$\Delta^a = (-1)^{\varepsilon_A} \frac{\delta_l}{\delta \phi^A} \frac{\delta}{\delta \phi_{Aa}^*} + (-1)^{\varepsilon_A+1} \varepsilon^{ab} \frac{\delta_l}{\delta \pi^{Ab}} \frac{\delta}{\delta \bar{\phi}_A}$$

coincides with the corresponding objects used in [4, 5]. The form of the integration measure in eq. (3.1)

$$d\Phi d\bar{\Phi} \rho(\bar{\Phi}) = d\phi d\phi^* d\pi d\bar{\phi} d\lambda dJ \delta(J)$$

and the component representation of the operator  $V^a$  in eq. (2.1)

$$V^a = \varepsilon^{ab} \phi_{Ab}^* \frac{\delta}{\delta \bar{\phi}_A} - J_A \frac{\delta}{\delta \phi_{Aa}^*}$$

imply that, when  $J_A = 0$ , the generating equation (3.4) for the action  $\mathcal{W} = W|_{J=0}$  coincides with the one used in [5] when formulating the rules of the modified triplectic approach. As for the equation used to define the gauge functional  $X$  (3.5), note, first of all, that the operator  $U^a$  (2.1), having the component representation

$$U^a = (-1)^{\varepsilon_A} \varepsilon^{ab} \lambda^A \frac{\delta_l}{\delta \pi^{Ab}} - (-1)^{\varepsilon_A} \pi^{Aa} \frac{\delta_l}{\delta \phi^A},$$

coincides, when  $\lambda^A = 0$ , with the operator  $U^a$

$$U^a = -(-1)^{\varepsilon_A} \pi^{Aa} \frac{\delta_l}{\delta \phi^A}, \quad (5.1)$$

used in the generating equation that determines the gauge in [5]. Further, note that the functional  $\bar{\Phi}\Phi$  in (3.7) is given by

$$\bar{\Phi}\Phi = \bar{\phi}_A \lambda^A + \phi_{Aa}^* \pi^{Aa} - J_A \phi^A.$$

Then we can see that the generating equation for the functional  $\mathcal{X} = X + \bar{\phi}_A \lambda^A$  has the form of eq. (3.5) with the truncated operator  $U^a$  (5.1), which is formally identical with

the generating equation of the modified triplectic approach [5]. As a consequence, the vacuum functional

$$Z = \int d\phi d\phi^* d\pi d\bar{\phi} d\lambda \exp \left\{ \frac{i}{\hbar} \left[ \mathcal{W}(\phi, \pi, \phi^*, \bar{\phi}) + \mathcal{X}(\phi, \pi, \phi^*, \bar{\phi}, \lambda) + \phi_{Aa}^* \pi^{Aa} \right] \right\}$$

is identical with the one used in [5], limited to the case when the action  $\mathcal{W}$  does not depend on the variables  $\lambda^A$ .

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## Appendix A

Consider a set of linearly independent Fermionic doublets  $D^a = (\Delta^a, U^a, V^a)$  subject to eqs. (2.2) and (2.3). Let us introduce two Bosonic objects  $C^{(1,2)}$  by the rule

$$\begin{aligned} C^1 &= \{U^1, \Delta^2\} = -\{U^2, \Delta^1\} \neq 0, \\ C^2 &= \{V^1, \Delta^2\} = -\{V^2, \Delta^1\} \neq 0. \end{aligned}$$

It is straightforward to check that  $C^{(1,2)}$  commute with  $D^a$ , and, consequently, also with each other

$$[C^{(1,2)}, D^a] = [C^1, C^2] = 0. \quad (\text{A.1})$$

In general, the objects  $C^{(1,2)}$  may be linearly dependent, i.e.

$$C^1 = C^2 = 0, \quad (\text{A.2})$$

$$C^1 = \alpha C^2 \neq 0, \quad (\text{A.3})$$

$$C^1 = 0, C^2 \neq 0, \quad (\text{A.4})$$

$$C^1 \neq 0, C^2 = 0 \quad (\text{A.5})$$

and linearly independent, i.e.

$$C^1 \neq 0, C^2 \neq 0, C^2 \neq \alpha C^1. \quad (\text{A.6})$$

The above possibilities imply, by virtue of eqs. (2.2), (2.3), (A.1), that the whole set of Fermions  $D^a$  and Bosons  $C^{(1,2)}$  generally spans five different Lie superalgebras: a six-dimensional one (A.2), three seven-dimensional ones (A.3)–(A.5), and an eight-dimensional one (A.6).

From eqs. (2.2), (2.3) and (A.1) it follows that each of the resulting superalgebras  $\mathcal{G}$  is nilpotent (see, e.g., [16]) with respect to the supercommutator  $[ , ]$ , namely, for the sequence  $\mathcal{G}^{[i]}$

$$[\mathcal{G}, \mathcal{G}] = \mathcal{G}^{[1]}, [\mathcal{G}, \mathcal{G}^{[i-1]}] = \mathcal{G}^{[i]}, \mathcal{G} \equiv \mathcal{G}^{[0]}, i \geq 1$$

there exists an integer  $n$  such that  $\mathcal{G}^{[n]} = \{0\}$ . Thus, in the case (A.2), we have  $n = 1$ , while in the other cases (A.2)–(A.6),  $n = 2$ . On the other hand, all the given superalgebras possess a non-trivial ideal [16], namely, there exists a non-trivial subalgebra  $\mathcal{I}$

$$\mathcal{I} \subset \mathcal{G}, \mathcal{I} \neq \{0\}, \mathcal{I} \neq \mathcal{G}, [\mathcal{I}, \mathcal{I}] \subset \mathcal{I}$$

such that  $[\mathcal{G}, \mathcal{I}] \subset \mathcal{I}$ . Thus, for the superalgebra (A.2) the ideal  $\mathcal{I}$  is given by the linear span of any five elements from the set  $(U^a, V^a, \Delta^a)$ , and for the remaining cases (A.3)–(A.6), by the linear span of  $C^{(1,2)}$ . The existence of a non-trivial ideal  $\mathcal{I}$  implies that the above superalgebras do not belong to simple Lie superalgebras, described by the standard classification [16]. Note also that these superalgebras are not semi-simple [16] either, because the ideal  $\mathcal{I}$  is solvable, i.e. for the sequence  $\mathcal{I}^{(i)}$

$$[\mathcal{I}, \mathcal{I}] = \mathcal{I}^{(1)}, \quad [\mathcal{I}^{(i-1)}, \mathcal{I}^{(i-1)}] = \mathcal{I}^{(i)}, \quad \mathcal{I} \equiv \mathcal{I}^{(0)}, \quad i \geq 1$$

there exists an integer  $n$  such that  $\mathcal{I}^{(n)} = \{0\}$ . Thus, in all cases (A.2)–(A.6), we have  $n = 1$ .

Let us turn to the manifest representation of the operators  $U^a$  and  $V^a$  as generators of supertranslations (2.1) and consider the explicit choices  $\Delta_1^a, \Delta_2^a$  for the operator  $\Delta^a$  in the form (2.9), (2.10), respectively. It is straightforward to check that the cases (2.9) and (2.10) lead to different superalgebras. Thus, in the case (2.9) we have a realization of the form (A.3), with

$$C^1 = -C^2 = \int d^2\theta \frac{\delta_l}{\delta\Phi(\theta)} \frac{\partial^2}{\partial\theta^2} \frac{\delta}{\delta\bar{\Phi}(\theta)} (-1)^{\varepsilon_A}, \quad (\text{A.7})$$

while in the case (2.10) we arrive at the realization (A.4), with  $C^2$  given by (A.7). Finally, note that non-vanishing linear combinations of the operators  $\Delta_1^a, \Delta_2^a$  are restricted to the seven-dimensional superalgebras (A.3)–(A.5).

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